

# Chern-Simons theory and three-dimensional surfaces

Jemal Guven

*Instituto de Ciencias Nucleares  
Universidad Nacional Autónoma de México  
Apdo. Postal 70-543, 04510 México, DF, MEXICO*

## Abstract

There are two natural Chern-Simons theories associated with the embedding of a three-dimensional surface in Euclidean space; one is constructed using the induced metric connection – it involves only the intrinsic geometry, the other is extrinsic and uses the connection associated with the gauging of normal rotations. As such, the two theories appear to describe very different aspects of the surface geometry. Remarkably, at a classical level, they are equivalent. In particular, it will be shown that their stress tensors differ only by a null contribution that neither transmits force nor carries momentum. Their Euler-Lagrange equations provide identical constraints on the normal curvature. A new identity for the Cotton-York tensor is associated with the triviality of the Chern-Simons theory for embedded hypersurfaces implied by this equivalence. The corresponding null surface stress capturing this information will be constructed explicitly.

November 15, 2006

PACS: 04.60.Ds

## 1 Introduction

It is possible to formulate various geometrical properties of surfaces in terms of conserved ‘null’ surface stresses. These are stresses that do not transmit physical forces or, if the model is dynamical, carry momentum. They are conserved for geometrical reasons. As a result of this, their conservation possesses a validity transcending the specific physical model in which they appear. On the other hand, the appearance of null stresses within a model will often convey information that is central to its understanding. One particular physical model illustrates this point very nicely.

Consider a three-dimensional surface in an ambient Euclidean (or Minkowski) space. In the spirit of Deser, Jackiw and Templeton’s work on three dimensional gravity in the early eighties, one may associate a Chern-Simons theory with the intrinsic geometry of this surface [1]. In

arXiv:hep-th/0611144 v1 14 Nov 2006

contrast to their model, where the dynamical variables are provided by the metric, in a theory of surfaces the relevant metric is the one induced by the embedding of the surface in space. Thus the appropriate independent dynamical variables are the functions which describe this embedding. This difference is reflected in the Euler-Lagrange equations: whereas the Euler-Lagrange derivative with respect to the metric is given by the Cotton-York tensor [2], its counterpart for an embedded surface is a contraction of this tensor with the extrinsic curvatures.

If there are sufficient extra dimensions this distinction is not important: it is possible to peel off the curvatures and recover the model for gravity in a manner analogous to the recovery of Einstein gravity from Regge and Teitelboim's gravity à la string [3]. In general, however, the content of this equation depends sensitively on the number of extra dimensions. In particular, if the surface is embedded as a hypersurface, one can show that the contraction vanishes identically. There are thus no local degrees of freedom associated with this limit of the theory; the Chern-Simons theory induced by this embedding, unlike the theory based on a metric on which it is modeled, is trivial as a physical theory.

The Cotton-York tensor is associated with the intrinsic geometry of the surface. The vanishing of its contraction with the extrinsic curvature on a hypersurface is a consequence of the fact that the intrinsic geometry is constrained by its spatial environment. This property of the Cotton-York tensor is captured by the statement that the stress associated with a Chern-Simons theory of hypersurfaces is null. As such, it may be constructed as the curl of a potential. This potential will be constructed explicitly for the hypersurface limit of the Chern-Simons theory.

If the number of extra dimensions is increased the physics described by the Chern-Simons action is not trivial. It is possible to show that local degrees of freedom are associated with the twisting of the surface in a five or higher-dimensional space. This is characterized by the extrinsic twist or normal curvature associated with gauging the rotations of the normal vectors; it thus vanishes on a hypersurface. The Euler-Lagrange equations may be rewritten to reflect the fact that they provide a constraint on this curvature.

The normal curvature itself is a source of interesting geometrical invariants. In a recent paper, properties of the obvious Yang-Mills functional were described [4]. There are, however, other invariants that are peculiar to particular dimensions: one such invariant, characterizing two-dimensional surfaces embedded in four-dimensions, is the integral of the normal curvature itself; it was introduced by Polyakov in the context of a limit of QCD described by strings[5].

It is also evident that, for three dimensional surfaces, there is a Chern-Simons theory associated with the normal connection. This is a theory that appears to describe an aspect of the surface geometry that is unrelated to the theory based on the Christoffel connection. Again, the dynamical variables are the embedding functions of the surface, not the connection itself. A particularly effective way to examine this theory is to introduce auxiliary variables adapted to the connection [6] (as worked out in [4]). Remarkably, at a classical level, the two theories turn out to be equivalent. In particular, it will be shown that their stress tensors differ only by

a null contribution. Their Euler-Lagrange equations thus provide identical constraints on the normal curvature of  $\Sigma$ .

It is not too surprising that these two Chern-Simons theories are related. While the role played by the normal curvature is very different from the one played by the extrinsic curvature of the surface, it is not independent of the latter: just as the Gauss-Codazzi equations determine the intrinsic Riemann curvature, the Ricci integrability conditions determine the normal curvature completely in terms of a quadratic in the extrinsic curvature [7]. What is unexpected is that the two theories are identical. The challenge is to understand why.

## 2 Chern-Simons theory, hypersurfaces and null stresses

Consider a three-dimensional surface (worldsheet)  $\Sigma : u^A \rightarrow \mathbf{X}(u^A)$  embedded in a  $N + 3$ -dimensional Euclidean (Minkowski) space,  $\mathbf{X} = (X^1, \dots, X^{N+3})$ ,  $N \geq 1$ . Let some action  $S[\mathbf{X}]$  be associated with this surface. The response of  $S$  to a deformation of the surface  $\mathbf{X} \rightarrow \mathbf{X} + \delta\mathbf{X}$  is characterized by a stress ‘tensor’  $\mathbf{f}^A$ . The notation reflects the fact that  $\mathbf{f}^A$  form  $N + 3$  surface vector fields, one for each spatial extension. The invariance of  $S[\mathbf{X}]$  under ambient Euclidean motions (Lorentz transformations) implies that [8]

$$\delta S = - \int dV \mathbf{f}^A \cdot \partial_A \delta \mathbf{X}, \quad (1)$$

where  $dV$  is the volume element induced on  $\Sigma$ . The dot represents the usual Euclidean (Lorentz) invariant inner product on the background. A momentum density, given by  $l_A \mathbf{f}^A$ , may be associated with any unit (timelike) vector field,  $l_A$ , on  $\Sigma$ . On classical membrane trajectories,  $\mathbf{f}^A$  is covariantly conserved:

$$\nabla_A \mathbf{f}^A = 0, \quad (2)$$

where  $\nabla_A$  is the covariant derivative compatible with the induced metric  $g_{AB}$  on  $\Sigma$ .  $\mathbf{f}^A$  is the Noether current associated with the translational invariance of  $S$ . There is, however, an ambiguity inherent in the definition of  $\mathbf{f}^A$ . This is because, within the surface,  $\mathbf{f}^A$  is defined only modulo an additional ‘null’ stress  $\mathbf{h}^A$  of the form

$$\mathbf{h}^A = \epsilon^{ABC} \partial_B \mathbf{A}_C, \quad (3)$$

where  $\epsilon^{ABC}$  is the surface Levi-Civita tensor. The potential  $\mathbf{A}^A$  has the same structure as  $\mathbf{h}^A$ . Note that the conservation of  $\mathbf{h}^A$ ,  $\nabla_A \mathbf{h}^A = 0$ , does not depend on the specific metric induced on  $\Sigma$ . This is captured by the compatibility of the Levi-Civita tensor with  $\nabla_A$ ,  $\nabla_A \epsilon_{BCD} = 0$ , as well as the first Bianchi identity,  $\epsilon^{ABC} R_{ABC}{}^D = 0$ :

$$\nabla_A \mathbf{h}^A = \frac{1}{2} \epsilon^{ABC} [\nabla_A, \nabla_B] \mathbf{A}_C = \frac{1}{2} \epsilon^{ABC} R_{ABC}{}^D \mathbf{A}_D = 0. \quad (4)$$

For the action  $S[\mathbf{X}]$ , consider the Chern-Simons action associated with the intrinsic geometry of  $\Sigma$ ,

$$S_{\text{CS}}[A_A] = \int d^3u \varepsilon^{ABC} \text{Tr} \left( \frac{1}{2} A_A \partial_B A_C + \frac{1}{3} A_A A_B A_C \right), \quad (5)$$

where  $(A_A)^B{}_C = \Gamma_{AC}^B$ , and the  $\Gamma_{AC}^B$  are the Christoffel connections constructed with the induced metric  $g_{AB}$  on  $\Sigma$  [1]. The connection is thus itself a functional of  $\mathbf{X}$ :  $\Gamma_{AB}^C = \Gamma_{AB}^C[\mathbf{X}]$ .  $\varepsilon^{ABC}$  is the Levi-Civita density. Thus the metric dependence of  $S$  occurs only through  $\Gamma_{AC}^B$ .

It is well known that if  $S$  is treated as a functional of  $g_{AB}$ , then

$$\delta S_{\text{CS}} = - \int dV C^{AB} \delta g_{AB}, \quad (6)$$

where the symmetric

$$C^{AB} = \epsilon^{ACD} \nabla_C \left( \mathcal{R}_D{}^B - \frac{1}{4} g_D{}^B \mathcal{R} \right) \quad (7)$$

is the Cotton-York tensor, defined in terms of the Ricci tensor  $\mathcal{R}_{AB}$  and the scalar curvature  $\mathcal{R}$ . This tensor is thus identified as the metric stress tensor.  $C^{AB}$  is conformally invariant. It vanishes if (and only if) the geometry is flat. We are, however, interested in  $S$  not as a functional of  $g_{AB}$  but as a functional of  $\mathbf{X}$ . Thus  $g_{AB} = \mathbf{e}_A \cdot \mathbf{e}_B$  is the induced metric, where the  $\mathbf{e}_A = \partial_A \mathbf{X}$  are the tangent vector fields along the parameter curves. The deformation of the metric, in response to a deformation  $\mathbf{X} \rightarrow \mathbf{X} + \delta \mathbf{X}$  of the surface, is given by  $\delta g_{AB} = 2\mathbf{e}_{(A} \cdot \partial_{B)} \delta \mathbf{X}$ . Thus, the stress associated with  $S_{\text{CS}}$  is identified as

$$\mathbf{f}_{\text{CS}}^A = 2C^{AB} \mathbf{e}_B. \quad (8)$$

Let the  $N$  normal vectors be labeled  $\{\mathbf{n}^1, \dots, \mathbf{n}^N\}$ ; and suppose that  $\mathbf{n}^I \cdot \mathbf{n}^J = \delta^{IJ}$ . The Gauss equations

$$\nabla_A \mathbf{e}_B = -K_{AB}^I \mathbf{n}_I, \quad (9)$$

define  $N$  extrinsic curvature tensors  $K_{AB}^I$ ,  $I = 1, \dots, N$ . These tensors are symmetric.

Using Eq.(9) it is possible to expand  $\nabla_A \mathbf{f}_{\text{CS}}^A$  as a linear combination of the tangent vectors adapted to the surface,  $\{\mathbf{e}_A, \mathbf{n}^I\}$ :

$$\nabla_A \mathbf{f}_{\text{CS}}^A = 2\nabla_A C^{AB} \mathbf{e}_B - 2C^{AB} K_{AB}^I \mathbf{n}_I. \quad (10)$$

The conservation of the Cotton-York tensor,

$$\nabla_B C^{AB} = 0, \quad (11)$$

emerges as a consequence of the reparametrization invariance of  $S_{\text{CS}}$ . The Euler-Lagrange equations are captured by the vanishing of the normal component of  $\nabla_A \mathbf{f}_{\text{CS}}^A$

$$C^{AB} K_{AB}^I = 0. \quad (12)$$

It is useful to place Eqs.(11) and (12) in context. In general, if it is possible to express the action  $S[\mathbf{X}]$  as a functional of the induced metric and the Riemann tensor, the conserved stress tensor can be cast as  $\mathbf{f}^A = T^{AB} \mathbf{e}_B$ . The metric stress tensor  $T^{AB}$  is conserved,  $\nabla_A T^{AB} = 0$ , as a consequence of the surface reparametrization invariance of  $S$ . This is completely analogous to the conservation of the Einstein tensor  $G^{AB}$  which follows from the reparametrization invariance of the Hilbert-Einstein action. The equations  $T^{AB} K_{AB}^I = 0$ , however, describe the classical trajectories of surfaces. For example, in Regge-Teitelboim gravity, the Hilbert-Einstein action induced by the embedding describes a theory of four dimensional worldsheets. The vacuum Einstein equations are replaced by the equations,  $G^{AB} K_{AB}^I = 0$  [3].

Let us first look at a surface embedded as a (timelike) hypersurface with a single (spacelike) normal,  $\mathbf{n}$  and corresponding curvature  $K_{AB}$ . Then  $C^{AB} K_{AB}$  vanishes identically in Euclidean space.  $\mathbf{f}_{CS}^A$  is thus a null stress.

This identity involves the surface integrability conditions in an essential way. Pulling  $K_{AB}$  into the derivative appearing in Eq.(7), one has

$$C^{AB} K_{AB} = \epsilon^{ACD} \nabla_C \left[ \left( \mathcal{R}_D^B - \frac{1}{4} g_D^B \mathcal{R} \right) K_{AB} \right] - \epsilon^{ACD} \left( \mathcal{R}_D^B - \frac{1}{4} g_D^B \mathcal{R} \right) \nabla_C K_{AB}. \quad (13)$$

The Gauss-Codazzi equations

$$\mathcal{R}_{AB} = K K_{AB} - K_{AC} K^C_B \quad (14)$$

express the Riemann (or Ricci) tensor in terms of the extrinsic curvature. This permits a presentation of the tensor  $P_{AB} = (\mathcal{R}_B^C - g_B^C R/4) K_{AC}$  as a polynomial in  $K_{AB}$  and  $g_{AB}$ . As such, it is manifestly symmetric in  $A$  and  $B$ . Thus the first term vanishes. Also the Codazzi-Mainardi equations,

$$\nabla_A K_{BC} - \nabla_B K_{AC} = 0, \quad (15)$$

imply that  $\nabla_C K_{AB}$  is symmetric in  $A$  and  $C$ . Thus the second term also vanishes. One concludes that

$$C^{AB} K_{AB} = 0 \quad (16)$$

for any hypersurface.

There is thus no bulk response in  $S_{CS}$  to a deformation of  $\Sigma$  as a hypersurface.

By linking intrinsic geometry to its environment, this identity has no analogue in Deser, Jackiw and Templeton's metric framework. It is, however, still curious in view of its elementary nature that this identity does not feature in the literature. Although Chern-Simons theory has directed us to its existence, it is worth emphasizing that it is a purely geometrical identity, tying together the Gauss-Codazzi and the Codazzi-Mainardi equations (14) and (15) in a subtle way. It also raises a host of interesting questions of a geometrical nature. Eq.(16) is a very strong constraint: is  $C^{AB}$  the only tensor satisfying a relation of this form? It is clear that there is no

finite polynomial in  $K_{AB}$  itself that is ‘orthogonal’ to  $K_{AB}$ . Derivatives of  $K_{AB}$  are required. There is good reason to believe (see section 5) that in three-dimensions, the Cotton-York tensor appears to be the unique symmetric tensor involving first derivatives satisfying Eq.(16). It is simple to discount the existence of an analogous tensor for two-dimensional surfaces. It would be interesting to determine if an analogue exists in higher odd dimensions (where Chern-Simons theories exist). And it would be interesting to know if (higher derivative) counterparts exist for higher dimensional surfaces, even or odd.

For completeness, in the appendix, we will show how the null stress for a hypersurface may be reconstructed from a potential.

### 3 Adding co-dimensions

If  $\Sigma$  is not a hypersurface, Eq.(12) is no longer generally valid off shell. It is instructive to dismantle the equation using surface theory to understand its content.

The counterpart of the Gauss equations for the normal vectors is provided by the Weingarten equations [7]

$$\tilde{\nabla}_A \mathbf{n}^I = K_{AB}^I g^{BC} \mathbf{e}_C. \quad (17)$$

For co-dimensions higher than one, these equations involves the  $SO(N)$  covariant derivative  $\tilde{\nabla}_A$ , defined by [9]

$$\tilde{\nabla}_A \mathbf{n}^I = \nabla_A \mathbf{n}^I + \omega^I_{JA} \mathbf{n}^J, \quad (18)$$

where the normal connection  $\omega^{IJ}_A$  is given by

$$\omega^{IJ}_A = \mathbf{n}^I \cdot \partial_A \mathbf{n}^J = -\omega^{JI}_A. \quad (19)$$

$\omega^{IJ}_A$  may be identified with the  $SO(N)$  connection associated with the gauging of normal rotations. Its curvature, satisfying

$$[\tilde{\nabla}_A, \tilde{\nabla}_B] \mathbf{n}^I = \Omega_{AB}^I{}_J \mathbf{n}^J, \quad (20)$$

is given by

$$\Omega_{AB}^{IJ} = \partial_A \omega^{IJ}_B - \partial_B \omega^{IJ}_A + \omega^I_{KA} \omega^{KJ}_B - (A \leftrightarrow B). \quad (21)$$

The intrinsic and the extrinsic geometries of  $\Sigma$  are related by integrability conditions. The generalizations of the Gauss-Codazzi and Codazzi-Mainardi equations for a hypersurface are:

$$\mathcal{R}_{ABCD} - K_{ACI} K_{BD}^I + K_{ADI} K_{BC}^I = 0, \quad (22)$$

$$\tilde{\nabla}_A K_{BC}^I - \tilde{\nabla}_B K_{AC}^I = 0. \quad (23)$$

A third set of equations constrains  $\Omega_{ABIJ}$ :

$$\Omega_{ABIJ} - K_{ACI} K_{BJ}^C + K_{ACJ} K_{BI}^C = 0. \quad (24)$$

Thus the normal curvature  $\Omega_{AB}^{IJ}$ , like the Riemann curvature  $\mathcal{R}_{ABCD}$ , is determined completely by the extrinsic curvature,  $K_{AB}^I$ . If  $K_{AB}^I$  vanishes in all but one direction,  $\Omega_{ABIJ}$  will also vanish.

Now one has, replacing Eq.(13),

$$C^{AB} K_{AB}^I = \epsilon^{ACD} \tilde{\nabla}_C \left[ \left( \mathcal{R}_D^B - \frac{1}{4} g_D^B \mathcal{R} \right) K_{AB}^I \right] - \epsilon^{ACD} \left( \mathcal{R}_D^B - \frac{1}{4} g_D^B \mathcal{R} \right) \tilde{\nabla}_C K_{AB}^I. \quad (25)$$

The second term again vanishes identically because of the symmetry implied by the Codazzi-Mainardi equations (23). However, the first term does not generally vanish if  $N > 1$ . Instead, one possesses the identity

$$\begin{aligned} \left( \mathcal{R}_D^B - \frac{1}{4} g_D^B \mathcal{R} \right) K_{AB}^I &= K_D^{BJ} K_J K_{AB}^I - K_{DE}^J K^{EB}{}_J K_{AB}^I \\ &= \frac{1}{2} \left( \Omega_{AD}^{IJ} K_J - \Omega_{AE}^{IJ} K^E{}_{DJ} \right), \end{aligned} \quad (26)$$

where the first line follows from the Gauss-Codazzi equations (22), and the second from Eq.(24) on swapping normal partners. Thus, using the Bianchi identities,  $\nabla_{[C} \Omega_{AB]}^{IJ} = 0$ , one may express

$$C^{AB} K_{AB}^I = \frac{1}{2} \epsilon^{ACD} \left( \Omega_{AD}^{IJ} \tilde{\nabla}_C K_J - \tilde{\nabla}_C \Omega_{AE}^{IJ} K^E{}_{DJ} \right), \quad (27)$$

which generally does not vanish. The Euler-Lagrange equations Eqs.(12) thus place constraints on  $\Omega_{AB}^{IJ}$ . If  $\Omega_{AB}^{IJ}$  vanishes, as it will for a hypersurface, the right hand side of Eq.(27) vanishes.

It would also be interesting to examine Eq.(27) perturbatively about a hypersurface to obtain some physical intuition concerning its content.

## 4 Chern-Simons theory and the normal connection

The embedded analogue of Deser, Jackiw and Teitelboim's action is not the only Chern-Simons theory one can construct with the surface degrees of freedom. An alternative connection  $A_A$  in Eq.(5) is provided by the  $SO(N)$  connection  $\omega_A^{IJ}$ . In terms of the gauge groups, and what the connect represent, this is a very different action from the one based on  $\Gamma_{AC}^B$ :  $SO(N)$  replaces  $GL(3)$ . However, once again it must be remembered that the dynamical variables are not the connection itself but the embedding functions  $\mathbf{X}$ . Classically, the two describe the same theory.

First recall how the Chern-Simons action responds to a change in the connection. If  $\omega_A^{IJ} \rightarrow \omega_A^{IJ} + \delta \omega_A^{IJ}$ , then

$$\delta_\omega S_{CS} = \int dV \mathcal{E}^A{}_{IJ} \delta \omega_A^{IJ}, \quad (28)$$

where

$$\mathcal{E}^A{}_{IJ} = \epsilon^{ABC} \Omega_{BCIJ}. \quad (29)$$

If the connection were the dynamical variable, the equilibria would be provided by the flat connections. Which is not very interesting.

In order to obtain the variation of  $S_{CS}$  with respect to  $\mathbf{X}$ , we will exploit the method of auxiliary variables [6]. Following the adaptation of this method to accommodate a normal connection presented in [4]), we treat  $\omega_A^{IJ}$  as a set of variables that are independent of  $\mathbf{X}$ . The price one pays is that one must now introduce Lagrange multipliers to enforce the constraints that connect  $\omega_A^{IJ}$  to the geometry. In this approach, the induced metric  $g_{AB}$ , as well as the basis vectors  $\{\mathbf{e}_A, \mathbf{n}^I\}$ , are also treated as independent variables. Thus construct the functional  $S[\mathbf{X}, \mathbf{e}_A, \mathbf{n}^I, \omega_A^{IJ}, g_{AB}, \mathcal{E}_A^{IJ}, T^{AB}, \lambda^{IJ}, \lambda^A_I, \mathbf{f}^A]$  given by

$$S[\mathbf{X}, \dots] = S_{CS}[\omega_A^{IJ}] - \int dV \left[ \mathcal{E}_{IJ}^A (\omega_A^{IJ} - \mathbf{n}^I \cdot \nabla_A \mathbf{n}^J) + \frac{1}{2} T^{AB} (g_{AB} - \mathbf{e}_A \cdot \mathbf{e}_B) \right] \\ + \int dV \left[ \frac{1}{2} \lambda_{IJ} (\mathbf{n}^I \cdot \mathbf{n}^J - \delta^{IJ}) + \lambda^A_I (\mathbf{n}^I \cdot \mathbf{e}_A) + \mathbf{f}^A \cdot (\mathbf{e}_A - \partial_A \mathbf{X}) \right]. \quad (30)$$

The variables  $\mathcal{E}_A^{IJ}, T^{AB}, \lambda^{IJ}, \lambda^A_I$  and  $\mathbf{f}^A$  are Lagrange multipliers. In this formulation,  $\mathbf{X}$  appears only in the last term – the constraint identifying the tangent vectors as derivatives of the embedding functions. Variation with respect to  $\mathbf{X}$  reproduces Eq.(1) for the action if the multiplier is identified with the stress tensor.

The Euler-Lagrange equations for  $\omega_A^{IJ}$  reproduce Eq.(29); however, unlike the theory whose dynamical variables are provided by the connection, in general  $\mathcal{E}_{IJ}^a \neq 0$  on shell.

In general, the Euler-Lagrange equations for the tangent vectors  $\mathbf{e}_A$  provide an expansion for  $\mathbf{f}^A$  in terms of the basis adapted to the surface  $\{\mathbf{e}_A, \mathbf{n}^I\}$ :

$$\mathbf{f}^A = T^{AB} \mathbf{e}_B + \lambda^A_I \mathbf{n}^I. \quad (31)$$

In particular, the conservation law (2) may be decomposed into its tangential and normal parts by projection and using the Gauss-Weingarten equations (9), (17):

$$\nabla_A T^{AB} - \lambda^A_I K_A^{BI} = 0, \quad (32)$$

$$\nabla_A \lambda^A_I + T^{AB} K_{ABI} = 0. \quad (33)$$

The normal projections (33) are the unconstrained Euler-Lagrange equations for the action  $S_{CS}$  with respect to the embedding functions  $\mathbf{X}$ . The tangential projections (32) are satisfied off-shell, providing the Bianchi identities associated with the reparametrization invariance of  $S_{CS}$ .

Next, note that  $S_{CS}$  does not depend on the metric. Thus the variation with respect to the induced metric  $g_{AB}$  vanishes. As a consequence the metric stress tensor  $T^{AB}$  vanishes:  $T^{AB} = 0$ . This would spell triviality if the action did not depend also on the extrinsic curvature. It does, however, have unexpected consequences. Note, in particular, that the stress is purely



normal, which would be impossible if the action depended only on  $g_{AB}$  and  $K_{AB}^I$  [6]. In this context, the vanishing tangential component also implies conformal invariance.

The normal component of  $\mathbf{f}^A$  is provided by the Lagrange multipliers,  $\lambda^A_I$ . To determine  $\lambda^A_I$ , consider the variation with respect to the normal vectors  $\mathbf{n}^I$ :

$$\lambda^A_{IJ} \nabla_A \mathbf{n}^J + \nabla_A (\mathcal{E}^A_{IJ} \mathbf{n}^J) + \lambda_{IJ} \mathbf{n}^J + \lambda^A_I \mathbf{e}_A = 0. \quad (34)$$

The tangential projections of this equation identify  $\lambda^A_I$ :

$$\lambda^A_I = -2\lambda^B_{IJ} K^A_B{}^J = -2\mathcal{E}^B_{IJ} K^A_B{}^J = -2\epsilon_{ABC} F^{BC}_{IJ} K^A_B{}^J. \quad (35)$$

The normal projections plays no role in the conservation law; for completeness, note that they identify the remaining Lagrange multiplier  $\lambda_{IJ}$ ,

$$\lambda_{IJ} = 2\omega_{A(I}{}^K \lambda^A_{|K|J)}, \quad (36)$$

as well as confirming the vanishing of the divergence of  $\mathcal{E}^A_{IJ}$ :

$$\tilde{\nabla}_A \lambda^A_{IJ} = \tilde{\nabla}_A \mathcal{E}^A_{IJ} = 0. \quad (37)$$

These are the Bianchi identities associated with the  $O(N)$  invariance of the Chern-Simons action.

One concludes that the stress takes the form

$$\mathbf{f}^A = -2K^A_B{}^J \epsilon^{BCD} \Omega_{CDIJ} \mathbf{n}^I. \quad (38)$$

The Euler-Lagrange equations are given by

$$-2\tilde{\nabla}_A (K^A_B{}^J \epsilon^{BCD} \Omega_{CDIJ}) = 0. \quad (39)$$

A straightforward calculation indicates that Eqs.(39) coincide with Eqs.(12). It can be shown that the difference between the two stress tensors is null. Remarkably, two apparently very different stresses, one normal, the other tangential describe the same theory.

The tangential projections of the conservation law for  $\mathbf{f}^A$  (32) reads

$$K^A_B{}^J \epsilon^{BCD} F_{CDIJ} K^I_{AE} = 0. \quad (40)$$

These equations are identically satisfied. One way to show this is to expand  $F_{CDIJ}$  in terms of  $K^I_{AB}$  using Eq.(24); one then re-pairs the quadratics in  $K^I_{AB}$  in terms of the Riemann tensor using Eq.(22). The resulting quadratic in the Riemann tensor is now expressed in terms of the Ricci tensor using the three-dimensional identity,

$$\mathcal{R}_{ABCD} = \left( \mathcal{R}_{A[C} - \frac{1}{4} g_{A[C} \mathcal{R} \right) g_{B|D]} + g_{A[C} \left( \mathcal{R}_{|B|D]} - \frac{1}{4} \mathcal{R} g_{|B|D]} \right). \quad (41)$$

The identity follows.

Finally, note that there are no non-trivial interpolations involving a mixing of intrinsic and extrinsic geometry. For if one sets  $(A_A)^C{}_B{}^I{}_J = \Gamma_{AB}^C \delta^I{}_J + \omega_{AJ}^I \delta^C{}_B$ , then  $S_{CS}[A_A] = S_{CS}[\Gamma] + S_{CS}[\omega]$ ; all cross terms involving  $\Gamma_{AB}^C$  and  $\omega_{AJ}^I$  vanish.

## 5 Conclusions

Chern-Simons theories of three-dimensional surfaces possess a number of intriguing properties. It has been shown that two apparently unconnected Chern-Simons theories – one associated with the induced Christoffel connection, the other with the normal connection – describe the same theory at a classical level. Their differences have been quantified in terms of a null stress that does not carry momentum or transmit forces. A remarkably simple geometrical property of hypersurfaces is associated with the triviality of the corresponding Chern-Simons theory: the stress is null. These results are also potentially relevant to Deser, Jackiw Templeton's original metric. A fuller examination of the connection is planned.

## Acknowledgments

Partial support from CONACyT grant 51111 as well as DGAPA PAPIIT grant IN119206-3 is acknowledged.

## Appendix: Null Potential for hypersurfaces

The Cotton-York tensor forms a tangential null stress for a hypersurface. Here it will demonstrated how to reconstruct this null stress from a potential.

Any null  $\mathbf{h}^A$  may be expanded with respect to the adapted basis

$$\mathbf{h}^A = h^{AB} \mathbf{e}_B + h^A \mathbf{n}. \quad (42)$$

It is reasonable to require that the tangential part of  $\mathbf{h}^A$  is symmetric, or equivalently, that  $\epsilon_{ABC} h^{BC} = 0$ . However,

$$\epsilon_{ABC} h^{BC} = \epsilon_{ABC} \epsilon^{Brs} \nabla_r \mathbf{A}_s \cdot \mathbf{e}^C = \nabla_A \mathbf{A}_B \cdot \mathbf{e}^B - \nabla_B \mathbf{A}_A \cdot \mathbf{e}^B. \quad (43)$$

Thus the symmetry of  $h^{AB}$  implies the constraint

$$\nabla_A \mathbf{A}_B \cdot \mathbf{e}^B - \nabla_B \mathbf{A}_A \cdot \mathbf{e}^B = 0 \quad (44)$$

on the potential,  $\mathbf{A}_A$ . This constraint simplifies if  $\mathbf{A}_A$  is tangential. Let  $\mathbf{A}^A = A^{AB} \mathbf{e}_B$ . Then Eq.(44) reads

$$\nabla_B (A^{AB} - g^{AB} A) = 0, \quad (45)$$

where  $A = A^A_A$ . Clearly both  $g^{AB}$  and  $K^{AB} - g^{AB} K$  satisfy Eq.(45). However,  $\mathbf{h}^A = 0$  in both cases. The simplest non-vanishing choice originates in the potential

$$A^{AB} = \mathcal{R}^{AB} - \frac{1}{4} R g^{AB}. \quad (46)$$

quadratic in the extrinsic curvature. Eq.(45) is satisfied by this tensor because the Einstein tensor  $G^{AB} = \mathcal{R}^{AB} - \mathcal{R}g^{AB}/2$  is divergence-free. With this choice, we reproduce  $\mathbf{h}^A = C^{AB} \mathbf{e}_B$ . The vanishing of the normal component is a consequence of the fact that any tensor polynomial in  $K_{AB}$  (and  $g_{AB}$ ) is symmetric.

Note that there are no higher order tensors polynomial in  $K_{AB}$  consistent with Eq.(45) in three-dimensions. This is because  $g_{AB}$ ,  $K_{AB} - g_{AB}K$  and  $G^{AB}$  are the only conserved tensors polynomial in  $K_{AB}$  on a three dimensional surface. Any other solution will involve higher derivatives.

A consequence of the conformal invariance of the Chern-Simons action is that the Cotton-York tensor is traceless:  $C^A{}_A = 0$ . More generally, the trace of the tangential component of  $\mathbf{h}^A$  is a divergence:

$$h^A{}_A = \nabla_A (\mathbf{X} \cdot \mathbf{h}^A). \quad (47)$$

Thus, there is a conserved surface vector field  $V^A = C^{AB} \nabla_B (|\mathbf{X}|^2)$  associated with the conformal symmetry.

There are other interesting null stresses associated with hypersurfaces. In particular, it is possible to construct a null stress capturing the conformal invariance of the Cotton-York tensor. One possibility is

$$\mathbf{h}_1^A = C^{AB} \mathbf{f}_{0B}, \quad (48)$$

where  $\mathbf{f}_0^A$  is given by

$$\mathbf{f}_0^A = (\mathbf{X} \cdot \mathbf{e}^A) \mathbf{n} - (\mathbf{X} \cdot \mathbf{n}) \mathbf{e}^A. \quad (49)$$

It is straightforward to show that  $\mathbf{h}_1^A$  is null. One makes use of the identity

$$\nabla_B \mathbf{f}_{0A} = g_{AB} \mathbf{n} + K_B{}^C ((\mathbf{X} \cdot \mathbf{e}_A) \mathbf{e}_C - (A \leftrightarrow C)). \quad (50)$$

The tangential projection is anti-symmetric in  $A$  and  $B$  and so does not contribute to the divergence of  $\mathbf{h}^A$ .

This statement does not involve Eq.(16). In this sense, it is a weaker statement than the nullity of  $\mathbf{f}_{\text{CS}}^A$  which does. Indeed, a null stress analogous to  $\mathbf{h}_1^A$  will be associated with any conformally invariant tensor of the intrinsic geometry. Thus, for example, the Bach tensor  $B^{AB}$  associated with the conformally invariant Weyl squared action of a four-dimensional surface provides a null stress of this form even though  $B^{AB} K_{AB} \neq 0$ .

It would be interesting to know if the null stress defined by Eq.(48) stems from a variational principle. This issue will be pursued elsewhere.

## References

- [1] Deser S, Jackiw R and Templeton S 1982 *Phys. Rev. Lett.* **48** 975; 1982 *Ann. Phys.* **140** 372

- [2] York Jr J W 1973 *J. Math. Phys.* **14** 456
- [3] Regge T and Teitelboim C *Proceedings of the Marcel Grossman Meeting*, Trieste, Italy, (1975), ed. Ruffini R (North-Holland, Amsterdam, 1977) 77
- [4] Capovilla R and Guven J 2006 *Yang-Mills theory à la string* To appear in *Gen. Rel. Grav.*
- [5] Polyakov A 1986 *Nucl. Phys. B* **268** 406
- [6] Guven J 2004 *J. Phys. A: Math and Gen.* **37** L313
- [7] Spivak M *A Comprehensive Introduction to Differential Geometry. Vols. 1-5, Second Edition* (Publish or Perish, 1979)
- [8] Arreaga G, Capovilla R, and Guven J 2000 *Ann. Phys.* **279** 126
- [9] Guven J 1993 *Phys. Rev.* **D48** 5562